Change of Variables

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, $T : \mathbb{R}^n \to \mathbb{R}^n$ a differentiable injective transformation. The Jacobian of T is the matrix

$$(J_T(\vec{u}))_{i,j} = \frac{\partial T_i}{\partial u_j}.$$

That is

$$J_T = \left\langle \frac{\partial T}{\partial u_1}, \ldots, \frac{\partial T}{\partial u_n} \right\rangle = \begin{bmatrix} \nabla T_1 \\ \vdots \\ \nabla T_n \end{bmatrix}.$$

If S is a closed bounded region in $\mathbb{R}^n$ and $D = T(S)$, then

$$\underset{D}{\iint \cdots \int} f(\vec{x})\, dV = \underset{S}{\iint \cdots \int} f(T(\vec{u}))|J_T(\vec{u})|\, dV.$$

So how to chose the correct substitution $\vec{x} = T(\vec{u})$?

Change of Variable Strategies

1. Aim for a simpler region of integration. If D is oddly shaped, maybe S can be a rectangle.
2. Experiment with transformations $(x, y) \mapsto (u, v)$ and $(u, v) \mapsto (x, y)$.
3. If the integrand is complicated, let it suggest new variables. Eg the integrand $\sqrt{\dfrac{x-y}{x+y}}$ probably wants $u = x - y$ and $v = x + y$.

Example 16.7.1. Evaluate the integral $\underset{D}{\iiint} xy\, dV$; D is bounded by the planes $y - x$=0, $y - x = 2$, $z - y = 0$, $z - y = 1$, $z = 0$, and $z = 3$.

Example 16.7.2. Evaluate the integral $\iiint_D z\,dV$; D is bounded by the paraboloid $z = 16 - x^2 - 4y^2$ and the xy-plane.

Example 16.7.3. Let D be the solid bounded by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Find the volume of D.

Example 16.7.4. Evaluate $\iint_R \left(\frac{y-x}{y+2x+1}\right)^4 dA$, where R is the parallelogram bounded by $y-x=1$, $y-x=2$, $y+2x=0$, and $y+2x=4$.